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T.M.T. COOLEN

A SURVEY ON HILBERT SPACE METHODS FOR HOMOGENEOUS ELLIPTIC BOUNDARY VALUE PROBLEMS

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A survey on Hilbert space methods for homogeneous elliptic boundary value problems

by

T.M.T. Coolen

ABSTRACT

A survey on the Hilbert space approach to existence, uniqueness and regularity of solutions of homogeneous elliptic boundary value problems is given.

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PREFACE

There are a large number of books in which a detailed exposition of the Hilbert space approach of boundary value problems for elliptic differential equations is given. In contrast with these books, some of which are listed at the end of this report, the aim of this report is to meet the needs of the applied mathematician, who is interested in the general idea of this approach of elliptic boundary value problems, and wishes to see the connection between the concepts involved without having to go into all of the details of the proofs. Therefore, in this survey nearly all proofs are omitted, although references are given.

Sobolev spaces of functions defined in a region of $\mathbb{R}^n$ are introduced in section 1. Replacing the original boundary value problem by the problem of finding a function satisfying an equation expressed in terms of a bilinear form defined on a Sobolev space is the subject of section 2. Section 3 is devoted to the existence of a unique solution of this new problem, the so-called generalized boundary value problem. Section 4 deals with the regularity of solutions, i.e., with the question under which conditions a solution of the generalized problem is a "classical" solution. Section 5 shows us how other boundary conditions (Neumann, mixed) can be examined. In section 6 some remarks on the regularity of solutions of boundary value problems in a domain with non-smooth boundaries (having "corners") are made.

1. SOBOLEV SPACES

The concept of the partial derivative of a function is generalized. Functions having generalized derivatives up to a certain order can be shown to form a Hilbert space, the so-called Sobolev space of that order. The further part of this section is concerned with the description of some properties of these spaces, among which the fact that Sobolev spaces can be imbedded in other spaces is of great importance.

1.1. Standard notations. The usual notations will be employed. Throughout this report $x = (x_1,\dots,x_n)$ is a variable point in the real n-dimensional Euclidean space $\mathbb{R}^n$; the Euclidean length of x is $|x| = (x_1^2+\dots+x_n^2)^{\frac{1}{2}}$; $D_i = \partial/\partial x_i$, $D^p = D_1^{p_1}\dots D_n^{p_n}$ where $p_1,\dots,p_n$ are non-negative integers and $p = (p_1,\dots,p_n)$, $|p| = p_1+\dots+p_n$. Given an open set Ω in $\mathbb{R}^n$, we shall denote by $C^m(\Omega)$, $m=0,1,2,\dots,\infty$, the set of all complete valued functions that are continuous in Ω together with their first m derivatives, and by $C^m(\bar{\Omega})$ the set of functions that have this property uniformly in Ω. If the boundary $\partial\Omega$ of Ω is "sufficiently" smooth -it will be explained later on more precisely which conditions should be imposed on $\partial\Omega$-, then $C^m(\bar{\Omega})$ is exactly the set of functions of which the derivatives up to order m may be considered as continuous functions on $\bar{\Omega}$. The subset of $C^m(\Omega)$ of all functions having a compact support in Ω is denoted by $C_0^m(\Omega)$.

Now take $m = \infty$. The set $C_0^\infty(\Omega)$ can be regarded as a linear space. If one additionally introduces a certain topology on $C_0^\infty(\Omega)$, the topological vector space $\mathcal{D}(\Omega)$ is obtained. A continuous linear functional on $\mathcal{D}(\Omega)$ is called a *distribution* in Ω. Elements of $C_0^\infty(\Omega)$ are usually denoted by ϕ, and frequently the notation $\langle f,\phi\rangle$ is used when meaning that f is a distribution in Ω.

The norm of an element of the Hilbert space of square integrable functions in Ω, $L^2(\Omega)$, is defined as

$$\| f \|_{0,\Omega} = \left(\int_\Omega |f(x)|^2 \, dx\right)^{\frac{1}{2}}$$

and the inner product as

$$(f,g)_{0,\Omega} = \int_\Omega f(x)\,\overline{g(x)}\,dx.$$

Finally, let u be a function defined in Ω; its restriction to a region $\Omega_1 \subset \Omega$ is usually denoted by $u|_{\Omega_1}$.

1.2. Definitions. If the distributional derivative $v = D^p u$ belongs to $L^2(\Omega)$, it is said that v is the *p-th weak derivative* of u. Since in that case

$$\int_\Omega v\,\phi\,dx = \langle D^p u,\phi\rangle \overset{def}{=} (-1)^{|p|}\langle u,D^p\phi\rangle = (-1)^{|p|}\int_\Omega uD^p\phi\,dx,$$

one can also define the p-th weak derivative as the function $v \in L^2(\Omega)$ satisfying

$$(v,\phi)_{0,\Omega} = (-1)^{|p|}\,(u,D^p\phi)_{0,\Omega}\ .$$

Let m be a non-negative integer, and Ω an open set in $\mathbb{R}^n$. Then the *m-th Sobolev space* $H^m(\Omega)$ by definition consists of all functions u having weak partial derivations up to order m. The scalar product of $H^m(\Omega)$ is defined as

$$(u,v)_{m,\Omega} = \sum_{|p|\le m}\int_\Omega D^p u D^p \bar{v}\,dx.$$

The norm associated with this scalar product is written as $\|u\|_{m,\Omega}$. When confusion is unlikely, the index Ω is often suppressed.

Notice that $H^0(\Omega) = L^2(\Omega)$. From the definitions it follows that

$$C_0^\infty(\Omega) \subset H^k(\Omega) \subset H^m(\Omega) \subset L^2(\Omega) \qquad \text{if } k \ge m \ge 0,$$

where the inclusions should be taken in the set theoretical sense. The following theorem is not hard to prove.

1.3. Theorem. The space $H^m(\Omega)$ as defined above is a Hilbert space in the norm $\|\cdot\|_{m,\Omega}$.

Proof: See Wloka [1969], p. 2. □

1.4. <u>Remark</u>. An alternative definition for $H^m(\Omega)$ can be given in the following manner. Let $C^{m*}(\Omega)$ denote the linear normed space of all functions u in $C^m(\Omega)$ having finite norm $\|u\|_{m,\Omega}$. A sequence $(u_k)_{k=1}^{\infty}$ is a Cauchy sequence in $C^{m*}(\Omega)$, if

$$\| D^p u_k - D^p u_\ell \|_{0,\Omega} \to 0 \qquad \text{as } k,\ell\to\infty,\ 0 \le |p| \le m.$$

where now $D^p u_k$ means the derivative in a classical sense. Since $L^2(\Omega)$ is a complete space, $D^p u_k$ converges to a certain $u^{(p)} \in L^2(\Omega)$. One is accustomed to call $u^{(p)}$ the *p-th strong derivative* of u. The m-th Sobolev space is then defined to consist of all functions $u \in L^2(\Omega)$ for which there exists a Cauchy sequence (u_k) in $C^{m*}(\Omega)$ such that $\| u-u_k \|_{0,\Omega} \to 0$ as $k \to \infty$.

For very general domains Ω, the two definitions are equivalent. Friedman [1969], p. 14 ff., gives a proof in case Ω is bounded, Agmon [1965], p. 11 ff., in case Ω satisfies a certain, easily fulfilled, condition.

Note, that once this equivalence is established, $H^m(\Omega)$ can be seen as the completion of $C^{m*}(\Omega)$ with respect to the norm $\|\cdot\|_{m,\Omega}$, and even of $C^{\infty,m*}(\Omega)$, by which we mean the linear normed space of all functions u of $C^\infty(\Omega)$ with finite norm $\|u\|_{m,\Omega}$. We shall be in need of a classification of the types of boundaries an open set in $\mathbb{R}^n$ can have. First of all there is the rather general cone condition. Besides we have more specific smoothness conditions.

1.5. <u>Definition</u>. (See figure 1.1). An open set $\Omega \subset \mathbb{R}^n$ is said to satisfy the *cone condition* if there exists a fixed spherical cone C with a certain opening and height, such that for each $x \in \Omega$ one can construct a cone C' with vertex x congruent to C that lies completely within Ω.

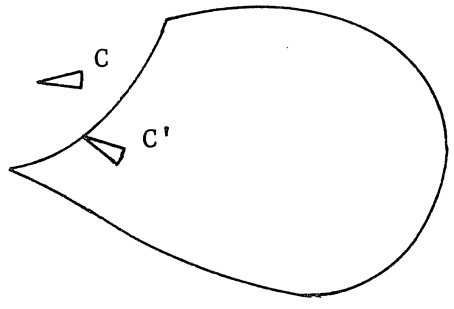


Figure 1.1

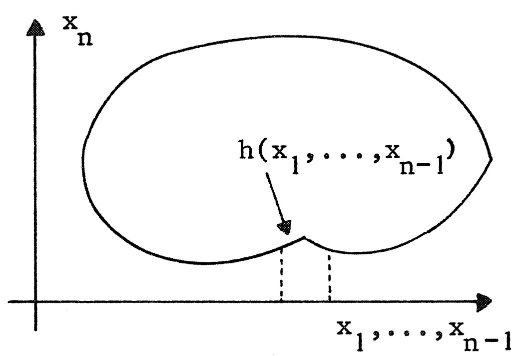


Figure 1.2

1.6. Definition. The boundary $\partial\Omega$ of a domain $\Omega \subset \mathbb{R}^n$ is said to be of *class* C^k *in the neighbourhood of* $x^0 \in \partial\Omega$, if there exists an open set Ω_0 (e.g. a ball), with $x^0 \in \Omega_0$, such that each point x of $\partial\Omega \cap \Omega_0$ can be represented in the form

$$(1.1)\qquad x = (x_1,\ldots,x_{i-1},h(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n),x_{i+1},\ldots,x_n),$$

and that

$$(1.2)\qquad x_i > h(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$$

in $\Omega_0 \cap \Omega$, both for some i, $1 \le i \le n$, with h a k times continuously differentiable function. Note that, in particular, $x_i^0 = h(x_1^0,\ldots,x_{i-1}^0,x_{i+1}^0,\ldots,x_n^0)$. In figure 1.2 the situation is shown for i = n.

If h is k times differentiable, and if the derivatives of order k are Lipschitz continuous, $\partial\Omega$ is said to be of class $C^{k,1}$ in the neighbourhood of $x \in \partial\Omega$.

If the boundary is of class C^k ($C^{k,1}$) in a neighbourhood of each $x \in \partial\Omega$, then one simply says that $\partial\Omega$ is of class C^k ($C^{k,1}$).

Clearly, the direction of the normal to the boundary is only defined in points $x \in \partial\Omega$ in the neighbourhood of which $\partial\Omega$ is of class C^1. Furthermore, if $\partial\Omega$ is of class $C^{0,1}$, Ω has the cone property.

We now wish to generalize the concept of zero values of the derivatives taken along the normal to the boundary of a certain function defined on Ω. To this end we first give a definition.

1.7. Definition. $H_0^m(\Omega)$ is the completion of $C_0^\infty(\Omega)$ in $H^m(\Omega)$. It is well-known that $C_0^\infty(\Omega)$ is dense in $L^2(\Omega)$, from which it follows that $H_0^0(\Omega) = L^2(\Omega)$. For $m \ge 1$, $H_0^m(\Omega)$ is a real subset of $H^m(\Omega)$.

1.8. Lemma. Let x_0 be a point on the boundary of a domain Ω, and let $\partial\Omega$ be Lipschitz continuous in a neighbourhood of x_0. If $u \in H_0^m(\Omega)$, if $D^p u$ is continuous in a neighbourhood of x_0 in Ω, and if $|p| \le m-1$, then $D^p u$ vanishes at x_0.

Proof: See Agmon [1965], p. 106 ff. □

A direct consequence of this lemma is that if $\partial\Omega$ is Lipschitz continuous, if $u \in H_0^m(\Omega) \cap C^{m-1}(\bar{\Omega})$, then $D^p u = 0$ for all p with $|p| \leq m-1$.

1.9. Lemma. If Ω is bounded and $\partial\Omega$ of class C^m, if $u \in C^m(\bar{\Omega})$ and if $D^p u = 0$ for $|p| \leq m-1$, then $u \in H_0^m(\Omega)$.

Proof: See Agmon [1965], p. 130. □

1.10. Definition. If $\partial\Omega$ is of class C^1, the normal to the boundary $\partial\Omega$ exists everywhere. In that case for functions $u \in C^m(\bar{\Omega})$ the property $D^p u = 0$ at $\partial\Omega$ for all p with $|p| \leq m-1$ is equivalent with $\partial^j u/\partial\nu^j = 0$ at $\partial\Omega$ for $j=0,\dots,m-1$, where $\partial/\partial\nu$ denotes the derivative along the inward normal to $\partial\Omega$. Lemmas 1.8 and 1.9 thus justify the following definition. If $u \in H_0^m(\Omega)$, then u is said to satisfy the homogeneous boundary conditions

$$\partial^j u/\partial\nu^j = 0, \qquad j=0,\dots,m-1$$

in the generalized sense. In the next section we shall see how the concept of generalized boundary conditions is used in connection with boundary value problems.

Now consider the identity mapping $H^k(\Omega) \to H^m(\Omega)$, $k > m \geq 0$. It is easily seen that this mapping is continuous. This is a very simple example of an imbedding theorem. The following theorem, due to *Rellich*, states a stronger result.

1.11. Theorem. Let Ω be a bounded domain in $\mathbb{R}^n$. Then the identity mapping $I: H_0^k(\Omega) \to H_0^m(\Omega)$ is compact for any $k > m \geq 0$. If the boundary of Ω is Lipschitz continuous, the identity mapping $H^k(\Omega) \to H^m(\Omega)$ is also compact for $k > m \geq 0$.

Proof: See Agmon [1965], p. 29 ff. and p. 99, Friedman [1969], p. 31. □

Often we wish to bring the m-th Sobolev norm of a function in connection with the traditional sup-norm. This is done by means of the next theorem.

1.12. Theorem. Let Ω be a bounded domain in $\mathbb{R}^n$ satisfying the cone condition. If u is a function in $C^m(\Omega) \cap H^m(\Omega)$, and if $m > n/2$, then

$$\sup_{x\in\Omega} |u(x)| \le \text{const.}\ \|u\|_{m,\Omega},$$

where the constant only depends on n and the opening and height of the cone.

Proof: Friedman [1969], p. 22. □

This section is concluded by an important imbedding theorem, that tells us in which cases a function having generalized derivatives can be considered as a function with classical derivatives.

1.13. Theorem. (*Sobolev*) Let $\Omega \subset \mathbb{R}^n$ be a bounded domain having the cone property, and let u belong to $H^m(\Omega)$ for some m. If $k \ge 0$ satisfies the inequality $m > k+n/2$, then u can be redefined on a set of measure zero in such a way that it belongs to $C^k(\Omega)$. One usually expresses this fact less rigorously by saying that for $m > k+n/2$ $H^m(\Omega) \subset C^k(\Omega)$.

Proof: Friedman [1969], p. 30, Agmon [1965], p. 32. □

Let us introduce for $u \in C^k(\Omega)$ the norm

$$(1.4) \qquad |||u|||_{k,\Omega} = \sum_{|p|\le k} \sup_{x\in\Omega} |D^p u(x)|$$

and the normed linear space $C^k_*(\Omega)$ consisting of all u in $C^k(\Omega)$ with finite norm (1.4). From theorem 1.12 it can be easily concluded that for $m > k+n/2$ the mapping $u \mapsto u$ defines a bounded imbedding of $H^m(\Omega)$ into $C^k_*(\Omega)$. Since for bounded domains Ω with boundary belonging to class $C^{0,1}$ there holds $C^k_*(\Omega) = C^k(\bar{\Omega})$, theorem 1.14 has the following corollary.

1.14. Corollary. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and its boundary $\partial\Omega$ of class $C^{0,1}$. If $m > k+n/2$, then $H^m(\Omega) \subset C^k(\bar{\Omega})$; moreover, the imbedding $H^m(\Omega) \to C^k(\bar{\Omega})$ defined by the identity mapping is continuous.

2. THE GENERALIZED DIRICHLET PROBLEM

The essential feature of Hilbert space methods for elliptic partial differential equations is to translate the boundary value problem into a so-called generalized problem of finding the solution of an equation in a Hilbert space expressed in terms of a bilinear form defined on that Hilbert space. Afterwards one then shows in which circumstances the solution of the generalized problem is also a solution to the problem originally stated. In this section we shall give a formulation for the Dirichlet problem for elliptic equations. In section 5 other boundary conditions are considered.

2.1. Definition. The differential operator

$$(2.1)\qquad L = -\sum_{i,j=1}^{n} D_i(a_{ij}(x)D_j\cdot) + \sum_{i=1}^{n} a_i(x)D_i\cdot + a_0(x)\cdot,$$

where the coefficients a_{ij} are real or complex valued, and, for the moment, are supposed to be sufficiently smooth for the operator to exist, is called *uniformly strongly elliptic* in an open set $\Omega \subset \mathbb{R}^n$, if there exists a constant $E > 0$ such that

$$(2.2)\qquad \mathrm{Re} \sum_{i,j=1}^{n} a_{ij}(x)\xi_i\xi_j \geq E \sum_{i=1}^{n} |\xi_i|^2 = E\,|\xi|^2$$

for all $\xi \in \mathbb{R}^n$ and $x \in \Omega$. The constant E is called the *module of ellipticity*.

2.2. Dirichlet problem. We now give a description of the Dirichlet problem for a second order elliptic differential operator. Consider a bounded domain $\Omega \subset \mathbb{R}^n$ with continuous boundary $\partial\Omega$. Let L be a uniformly strongly elliptic operator in Ω, and let f and g be given functions defined in Ω and on $\partial\Omega$ respectively. The Dirichlet problem then consists of finding a function u satisfying

$$(2.3)\qquad Lu = f \quad \text{in } \Omega,$$

$$(2.4)\qquad u = g \quad \text{on } \partial\Omega.$$

If a function u belongs to $C^2(\Omega)$, and satisfies (2.3), it is called a *classical solution* of the equation (2.3). A function $u \in C^0(\bar{\Omega})$ is said to satisfy the boundary condition (2.4) in the classical sense. If $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$ and if u satisfies (2.3) and (2.4), then one calls u a classical solution of the Dirichlet problem (2.3), (2.4).

In this stage of the exposition we wish to consider homogeneous boundary conditions only. This would be no loss of generality, if we had a theorem establishing the existence of a (sufficiently smooth) function F defined on $\bar{\Omega}$, satisfying (2.4) on $\partial\Omega$, since in that case we could consider w = = u-F, being a solution of the Dirichlet problem with vanishing boundary conditions

$$
\begin{aligned}
Lw &= \tilde{f} \overset{\text{def}}{=} f - LF \quad \text{in } \Omega, \\
w &= 0 \qquad\qquad\quad\;\; \text{on } \partial\Omega.
\end{aligned}
$$

It can be proved that such a function exists if Ω and $\partial\Omega$ satisfy certain conditions. Friedman [1969] gives a proof for the simple case one obtains if one imposes rather strong conditions on $\partial\Omega$. A general theory is found in Lions and Magénes [1968], or Nečas [1967]. Here, we merely postulate the existence of such a function F.

2.3. Bilinear forms. We wish to discuss in a more or less heuristic fashion the introduction of bilinear forms in connection with elliptic differential equations. Let us consider the Dirichlet problem with homogeneous boundary data

(2.3) $\quad Lu = f \quad \text{in } \Omega,$

(2.5) $\quad u = 0 \quad \text{on } \partial\Omega,$

and let u be a classical solution of this problem. Then for all $\phi \in C_0^\infty(\Omega)$

$$(Lu,\phi)_{0,\Omega} = (f,\phi)_{0,\Omega}.$$

By integration in parts we obtain

$$\Big(-\sum_{i,j=1}^{n} D_i a_{ij}(x) D_j u, \phi \Big)_0 = \sum_{i,j=1}^{n} (a_{ij}(x) D_j u, D_i \phi)_0 .$$

Hence, if u is a classical solution of equation (2.3), we have for all $\phi \in C_0^\infty(\Omega)$

$$\sum_{i,j=1}^{n} (a_{ij}(x) D_j u, D_i \phi)_0 + \sum_{i=1}^{n} (a_i(x) D_i u, \phi)_0 + (a_0(x), \phi), = (f, \phi)_0 \tag{2.6}$$

If we now define for u and v belonging to $C^1(\dot{\Omega})$ the expression B(u,v) as

$$B(u,v) = \sum_{i,j=1}^{n} (a_{ij}(x) D_j u, D_j v)_0 + \sum_{i=1}^{n} (a_i(x) D_i u, v)_0 + (a_0(x) u, v)_0 , \tag{2.7}$$

then we may write (2.6) as

$$B(u,\phi) = (f,\phi)_{0,\Omega} \qquad \text{for all } \phi \in C_0^\infty(\Omega). \tag{2.8}$$

It is clear that B(u,v) is a bilinear expression. It is called the *bilinear form associated with* L. Observe that it can be defined under less restrictive conditions on the coefficients a_{ij} than is possible in the definition of the operator L. In fact $a_{ij}(x) \in L^2(\Omega)$ is already sufficient. Moreover, if one allows generalized derivatives, then B(u,v) can be defined for all $u,v \in H^1(\Omega)$. In that case L in (2.1) is not a differential operator in the classical sense; but L does of course have a meaning in the distributional sense.

The *formal adjoint* of L is defined as

$$L^* = -\sum_{i,j=1}^{n} D_i(\bar{a}_{ij}(x) D_j \cdot) - \sum_{i=1}^{n} D_i(\bar{a}_i(x) \cdot) + a_0(x) \cdot \ . \tag{2.9}$$

Assuming the coefficients a_{ij} to be continuously differentiable, one easily obtains the equality

$$(L\phi,\psi)_{0,\Omega} = (\phi, L^*\psi)_{0,\Omega} \qquad \text{for all } \phi,\psi \in C_0^\infty(\Omega). \tag{2.10}$$

The right hand side of (2.10) also exists if one takes for ϕ a function $u \in L^2(\Omega)$, the left hand side does not. This leads us to the concept of weak solution: a function u is called a *weak solution* of Lu = f in Ω, if $u \in L^2(\Omega_0)$ for all Ω_0 with $\Omega_0 \subset \Omega$, and if

$$(2.11) \qquad (u,L^*\phi)_{0,\Omega} = (f,\phi)_{0,\Omega} \qquad \text{for all } \phi \in C_0^\infty(\Omega).$$

Here f is assumed to be square integrable. Some authors call u a *strong solution* of the equation Lu = f, if $u \in H^1(\Omega_0)$ for all Ω_0 with $\bar{\Omega}_0 \subset \Omega$, and if

$$(2.12) \qquad B(u,\phi) = (f,\phi)_{0,\Omega} \qquad \text{for all } \phi \in C_0^\infty(\Omega),$$

where B is the bilinear form associated with L. Since, by definition, $C_0^\infty(\Omega)$ is dense in $H_0^1(\Omega)$, (2.12) remains true for all $\phi \in H_0^1(\Omega)$.

Now the following assertion can be easily proved: let a_{ij}, a_i belong to $C^1(\Omega)$, a_0 belong to $C^0(\Omega)$, and let u be a function in $H^1(\Omega_0)$ for any Ω_0 with $\bar{\Omega}_0 \subset \Omega$, then u is a weak solution if and only if it is a strong solution. Therefore we are legitimated to forget the distinction between weak and strong in most cases, and to speak simply of generalized solutions of the equation Lu = f.

In the preceding section we have already given a generalized notion of the boundary condition (2.4). Additionally, we introduce the following terminology. The problem of finding a function $u \in H_0^1(\Omega)$, satisfying

$$B(u,v) = (f,v)_{0,\Omega} \qquad \text{for all } v \in H_0^1(\Omega),$$

where $f \in L^2(\Omega)$ is a given function, is called the *generalized problem for* (2.3), (2.4), or the *generalized Dirichlet problem* for homogeneous boundary data. A solution of the generalized problem is called a generalized solution of the Dirichlet problem (2.3), (2.4). In this terminology we state the following theorem.

2.4. Theorem. Let Ω be a bounded domain with continuous boundary, and let $a_{ij}, a_i \in C^1(\Omega)$, $a_0 \in C^0(\Omega)$. If u is a generalized solution of the Dirichlet problem for homogeneous boundary data, and if $u \in C^2(\Omega) \cap C^0(\bar{\Omega})$, then u is a classical solution.

Proof: From 2.3 it easily follows that u is a classical solution of the equation. That u satisfies the boundary condition (2.5) in the classical sense, is a consequence of theorem 1.10. □

2.5. Example. Consider in $\Omega \subset \mathbb{R}^n$ the Dirichlet problem

$$-\Delta u = f \quad \text{in } \Omega,$$
$$u = 0 \quad \text{on } \partial\Omega.$$

where Δ is the Laplace operator $\Sigma_{i=1}^{n} \partial^2/\partial x_i^2$; $-\Delta$ is elliptic with ellipticity module 1. The bilinear form associated with $-\Delta$ is the Dirichlet form (integral)

$$D(u,v) = \sum_{i=1}^{n} \int_\Omega D_i u D_i \bar{v}\, dx = \sum_{i=1}^{n} (D_i u, D_i v)_{0,\Omega}. \tag{2.13}$$

The generalized Dirichlet problem: find $u \in H_0^1(\Omega)$ such that

$$D(u,v) = (f,v)_{0,\Omega} \qquad \text{for all } v \in H_0^1(\Omega)$$

Now let Ω be bounded, and $\phi \in C_0^\infty(\Omega)$. Then there exists a cube $C: \{x \mid |x| < c\}$ such that $\Omega \subset C$. One has

$$\phi(x_1,\ldots,x_{n-1},x_n) = \int_{-c}^{x_n} D_n\phi(x_1,\ldots,x_{n-1},\xi_n)d\xi_n.$$

Applying the Cauchy-Schwarz inequality, one obtains

$$|\phi(x_1,\ldots,x_{n-1},x_n)|^2 \le 2c \int_{-c}^{c} |D_n\phi(x_1,\ldots,x_{n-1},\xi_n|^2\, d\xi_n.$$

Integration of this result with respect to x_n in the interval $(-c,c)$ yields

$$\int_{-c}^{c} |\phi(x_1,\ldots,x_{n-1},x_n)|^2\, dx_n \le 4c^2 \int_{-c}^{c} |D_n\phi(x_1,\ldots,x_{n-1},\xi_n)|\, d\xi_n.$$

Repeated integration with respect to the remaining n-1 variables leads to the result

(2.14) $$\|\phi\|^2_{0,\Omega} \leq 4c^2 \sum_{i=1}^{n} \|D_i\phi\|_{0,\Omega} \qquad \text{for all } \phi \in C_0^\infty(\Omega).$$

The above argument remains true if Ω is bounded in one direction, say x_n, only. Combining (2.13) and (2.14) one finds

(2.15) $$KD(u,u) \geq \|u\|^2_{0,\Omega} \qquad \text{for all } u \in H_0^1(\Omega),$$

which is known as *Poincaré's inequality*. One also has

(2.16) $$\|u\|^2_{1,\Omega} = D(u,u) + \|u\|^2_{0,\Omega} \leq (1+K)D(u,u)$$

for all $u \in H_0^1(\Omega)$, from which it follows that D(u,u) may be used as a norm equivalent to $\|u\|_1$.

2.6. Definition. Let V be a Hilbert space with norm $\|\cdot\|_V$, and let B(u,v) be a bilinear form defined on V. If, for some positive constant

$$\text{Re } B(u,u) \geq \text{const. } \|u\|_V^2 \qquad \text{for all } u \in V,$$

B is called *strongly coercive over* V.

In view of (2.16) D(u,v) is strongly coercive over $H_0^1(\Omega)$. We shall derive an inequality for bilinear forms which are associated to more general strongly elliptic operators.

2.7. Theorem. Let L be the uniformly strongly elliptic differential operator in $\Omega \subset \mathbb{R}^n$ of definition 2.1. Let further B be the bilinear form associated with L. Assume further that the coefficients $a_i(x)$ are uniformly bounded in Ω. Then there exist constants $c > 0$ and $\lambda_0 \geq 0$ such that

(2.17) $$\text{Re } B(u,u) \geq cE\|u\|_1^2 - \lambda_0\|u\|_0^2$$

for all $u \in H^1(\Omega)$, where the constant c only depends on n, and λ_0 only on n, E, and on sup $\{|a_i(x)| \mid x \in \Omega, i=0,\dots,n\}$.

Proof: Since the argument of the proof is both elementary and clarifying, we give it here. For simplicity we assume the coefficients and all functions occurring to be real. Using the uniformly strong ellipticity of L, we obtain

$$B(u,u) = \sum_{i,j=1}^{n} (a_{ij}D_j u, D_i u)_0 + \sum_{i=1}^{n} (a_i D_i u, u)_0 + (a_0 u, u)_0$$

$$\geq E \sum_{i=1}^{n} \|D_i u\|_0^2 - c_1 \left(\sum_{i=1}^{n} |(D_i u, u)_0| + \|u\|_0^2 \right),$$

where the positive constant c_1 is depending on the upperbound of $a_i(x)$, $i=0,1,\dots,n$. As for arbitrary $\varepsilon > 0$

$$2\,|(D_i u,u)_0| \leq 2\,\|D_i u\|_0 \|u\|_0 \leq \varepsilon\,\|D_i u\|_0^2 + \varepsilon^{-1}\,\|u\|_0^2,$$

it follows

$$B(u,u) \geq (E - c_1\varepsilon) \sum_{i=1}^{n} \|D_i u\|_0^2 - nc_1\varepsilon^{-1}\,\|u\|_0^2.$$

By an adequate choice of ε one obtains (2.17). □

Inequality (2.17) is a special case of the so-called *Gårding's inequality*. It expresses a somewhat weaker property than strong coercivity. In connection with this observation the following definition should be seen.

2.8. Definition. Let H be a Hilbert space with norm $\|\cdot\|_H$, and let V be a linear subspace of H that is dense in H. Suppose that V in its turn has a norm $\|\cdot\|_V$ in which it is a Hilbert space. Suppose further that $\|u\|_H \leq$ $\leq$ const. $\|u\|_V$ for all $u \in V$. A bilinear form B(u,v) defined for $u,v \in V$ is said to be *coercive over* V, if there exist constants $c_1 > 0$ and $c_2 \geq 0$ such that

(2.18) $$\operatorname{Re} B(u,u) \geq c_1 \|u\|_V^2 - c_2 \|u\|_H^2$$

for all $u \in V$.

Theorem 2.7 can be reformulated as follows: under the conditions given there $B(u,v)$ is coercive over $H^1(\Omega)$. An important special case of an elliptic operator L is considered in the next theorem.

2.9. Theorem. Let L be the strongly elliptic operator

$$L\cdot = -\sum_{i,j=1}^{n} D_i a_{ij}(x) D_j\cdot + a_0(x)\cdot,$$

where $a_0(x)$ satisfies for some α_0, $a_0(x) \geq \alpha_0 > 0$ for $x \in \Omega$. The bilinear form associated with L, i.e.

$$B(u,v) = \sum_{i,j=1}^{n} (a_{ij} D_j u, D_i v)_{0,\Omega} + (a_0 u, v)_{0,\Omega}$$

is strongly coercive over $H^1(\Omega)$.

Proof: The result follows from

$$\text{Re } B(u,u) \geq E \sum_{i=1}^{n} ||D_i u||_0^2 + \alpha_0 \; ||u||_0^2. \quad \Box$$

We have given a number of coerciveness results for bilinear forms associated with second order elliptic equations. The existence theory in the next section is based on the coerciveness of bilinear forms. But before proceeding to the existence of solutions we wish to generalize the above results for higher order elliptic equations.

2.10. Definition. The differential operator of order 2m

$$L = \sum_{|p|,|q|\leq m} (-1)^{|p|} D^p (a_{pq}(x) D^q \cdot) \tag{2.19}$$

is said to be uniformly strongly elliptic in a domain $\Omega \subset \mathbb{R}^n$, if there exists a constant $E > 0$ such that

$$\text{Re} \sum_{|p|=|q|=m} a_{pq}(x) \xi^{p+q} \geq E \; |\xi|^{2m}$$

for all $x \in \Omega$ and $\xi \in \mathbb{R}^n$. Notice that in (2.19) not the most general 2m-th order elliptic operator is defined. Nearly all authors restrict themselves to this type of operator.

2.11. Dirichlet problem for higher order equations. Let Ω be a bounded domain in $\mathbb{R}^n$, with continuously differentiable boundary. The Dirichlet problem for the uniformly elliptic operators of order 2m

$$(2.20) \qquad Lu = f \qquad \text{in } \Omega$$

is obtained by requiring

$$(2.21) \qquad \partial^j u/\partial\nu^j = g_j \quad \text{on } \partial\Omega,\ j=0,\dots,m-1,$$

where f and g_j, $j=0,\dots,m-1$ are given functions. In the same way as for second order elliptic equations we shall confine ourselves to the Dirichlet problem with homogeneous boundary data.

The introduction of bilinear forms is performed in the same fashion as was done for second order operators. Let ϕ be any function in $C_0^\infty(\Omega)$, then a classical solution u of (2.20) satisfies

$$(Lu,\phi)_{0,\Omega} = (f,\phi)_{0,\Omega}$$

Integrating the left hand side in parts, one finds

$$(2.22) \qquad \sum_{|p|,|q|\le m} (a_{pq}(x)D^q u, D^p\phi)_{0,\Omega} = (f,\phi)_{0,\Omega}$$

for all $\phi \in C_0^\infty(\Omega)$. Allowing generalized derivatives, one can define for all $u,v \in H^m(\Omega)$ the bilinear form associated with L

$$(2.23) \qquad B(u,v) = \sum_{|p|,|q|\le m} (a_{pq}(x)D^q u, D^p v)_{0,\Omega}.$$

The concept of generalized solution of the equation Lu = f, in case L is a 2m-th order elliptic operator, shows no difference with the case of second order equations. Exactly as one might expect, the operator

$$L^* = \sum_{|p|,|q|\le m} (-1)^{|p|} D^p(\bar{a}_{pq} D^q \cdot)$$

is called the *formal adjoint* of L (note that $L = L^*$ if the coefficients are real). If $u \in L^2(\Omega_0)$ for all Ω_0 with $\bar{\Omega}_0 \subset \Omega$, and if

$$(2.24) \qquad (u, L^*\phi)_{0,\Omega} = (f,\phi)_{0,\Omega} \qquad \text{for all } \phi \in C_0^\infty(\Omega),$$

then u is called a *weak solution*. And, as expected, u is called a *strong solution* if $u \in H^m(\Omega_0)$ for all Ω_0 with $\bar{\Omega}_0 \subset \Omega$, and if

$$B(u,\phi) = (f,\phi)_{0,\Omega} \qquad \text{for all } \phi \in C_0^\infty(\Omega),$$

where f is some given function belonging to $L^2(\Omega)$.

Just as in the second order case, one may overlook the distinction between weak and strong solution, because of the validity of the following assertion: let a_{pq} belong to $C^{|q|}(\Omega)$ for all multi-indices pq, and let u be a function of $H^m(\Omega_0)$ for any Ω_0 with $\bar{\Omega}_0 \subset \Omega$, then the concepts of weak and strong solutions are equivalent.

Finally, the problem of finding a function $u \in H_0^m(\Omega)$, such that

$$(2.25) \qquad B(u,v) = (f,v)_{0,\Omega}$$

holds for all $v \in H_0^m(\Omega)$, will be called the *generalized Dirichlet problem* with homogeneous boundary data for the equation $Lu = f$. Analogous to theorem 2.4 one has:

2.12. Theorem. Let Ω be a bounded domain with continuously differentiable boundary, and let $a_{pq}(x)$ belong to both $C^{|p|}(\Omega)$ and $C^{|q|}(\Omega)$. If u is a generalized solution of the Dirichlet problem for the equation $Lu = f$ with homogeneous boundary data, and if $u \in C^{2m}(\Omega) \cap C^{m-1}(\bar{\Omega})$, then u is a classical solution.

Proof: Combine the results of 2.11 and theorem 1.10. Observe that the C^1

boundary is only needed to be able to define the normal to the boundary in every point of $\partial\Omega$. □

2.13. Example. Consider in $\Omega \subset \mathbb{R}^n$ the Dirichlet problem

$$\begin{aligned} \Delta\Delta u &= f && \text{in } \Omega, \\ u = \partial u/\partial\nu &= 0 && \text{on } \partial\Omega, \end{aligned}$$

where Δ is the Laplace operator; $\Delta\Delta$ is called the *biharmonic* differential operator. To show that $\Delta\Delta$ is uniformly strongly elliptic, one must calculate the sum

$$\sum_{|p|=|q|=2} a_{pq}\xi^p\xi^q.$$

Since $a_{pq} = 1$ for $|p|=|q|=2$, $p = q$ and only one p_i in $p = (p_1,\ldots,p_n)$ is unequal to zero, $a_{pq} = 2$ for $|p|=|q|=2$, $p = q$ and only two p_i, p_j unequal to zero, and $a_{pq} = 0$ otherwise, one finds

$$\sum_{|p|=|q|=2} a_{pq}\xi^p\xi^q = |\xi|^4.$$

The bilinear form associated with $\Delta\Delta$ is

$$B(u,v) = (\Delta u,\Delta v)_{0,\Omega}, \tag{2.26}$$

which is defined for all $u,v \in H^2(\Omega)$.

In order to prove the strong coercivity over $H_0^2(\Omega)$ of this bilinear form, we need an inequality that is derived by repetitive application of (2.14)

$$\begin{aligned} \|\phi\|_2^2 &= \|\phi\|_0^2 + \sum_{i=1}^n \|D_i\phi\|_0^2 + \sum_{i,j=1}^n \|D_iD_j\phi\|_0^2 \\ &\le c_1 \sum_{i=1}^n \|D_i\phi\|_0^2 + \sum_{i,j=1}^n \|D_iD_j\phi\|_0^2 \le c_2 \sum_{i,j=1}^n \|D_iD_j\phi\|_0^2. \end{aligned}$$

Because the last quantity is equal to $\|\Delta u\|_0^2$, it is shown that

$$(2.27) \qquad B(u,u) = (\Delta u,\Delta u)_0 \geq c_3\|u\|_2^2 \qquad \text{for all } u \in H_0^2(\Omega).$$

The generalized Dirichlet problem with homogeneous boundary data is to find a $u \in H_0^2(\Omega)$ such that

$$(\Delta u,\Delta v)_{0,\Omega} = (f,v)_{0,\Omega} \qquad \text{for all } v \in H_0^2(\Omega).$$

The following theorem is merely a generalization of the argument establishing (2.27).

2.14. Theorem. Let L be a uniformly strongly elliptic operator with constant coefficients in Ω (Ω bounded in at least one direction), in which no derivatives of order less than 2m occur, i.e.

$$L\cdot = \sum_{|p|=|q|=m} a_{pq} D^{p+q}\cdot.$$

Then the bilinear form associated with L,

$$B(u,v) = \sum_{|p|=|q|=m} (a_{pq}D^p u, D^q v)_{0,\Omega}$$

is strongly coercive over $H_0^m(\Omega)$.

Proof: See also Agmon [1965], p. 80, or Friedman [1969], p. 34 ff. □

The generalization of theorem 2.7 for higher order elliptic operators is far more difficult to prove. Here we just state the result. It is known as *Gårding's inequality*.

2.15. Theorem. Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, and let the operator L given by (2.19) satisfy the following conditions:

(L1) L is uniformly strongly elliptic in Ω with a module of ellipticity E;

(L2) the coefficients $a_{pq}(x)$ of L are uniformly bounded in Ω;

(L3) the principal coefficients ($|p|=|q|=m$) are uniformly continuous in Ω.

Then the bilinear form B associated with L is coercive over $H_0^m(\Omega)$, i.e. there exist constants $c > 0$ and $\lambda_0 \geq 0$ such that

$$(2.28) \qquad \operatorname{Re} B(u,u) \geq cE\|u\|_{m,\Omega}^2 - \lambda_0\|u\|_{0,\Omega}^2$$

for all $u \in H_0^m(\Omega)$. Here, c only depends on m and n, and λ_0 on n,m,E, on the largest modulus of continuity of the principal coefficients and on $\sup\{|a_{pq}(x)| \mid x \in \Omega, |p|,|q| \leq m\}$.

Proof: See Agmon [1965], p. 75 ff. and Friedman [1969], p. 34 ff. The preceding theorem is a first step of the proof of Gårding's inequality. □

2.16. Remarks. Theorem 2.14 also applies to second order operators. Notice the difference between the theorem on coercivity above. In the second order case coercivity over $H^1(\Omega)$ is proved, but for higher order equations one has a less strong result, namely coercivity over $H_0^m(\Omega)$. Theorem 2.9, on a special second order operator, has no extension for $m > 1$. Finally, we have seen that if the principal coefficients are constants, and the other zero, the associated bilinear form is strongly coercive over $H_0^m(\Omega)$ for both $m = 1$ and $m > 1$.

3. EXISTENCE OF SOLUTIONS

The theory of existence of a solution of the generalized Dirichlet problem with homogeneous boundary data has its roots in two fundamental results, the Riesz representation theorem and the Fredholm alternative for compact operators in a Hilbert space. In this section first an extension of Riesz's theorem is given, known as the Lax-Milgram lemma, which is sufficient in case the bilinear form considered is *strongly* coercive. If the bilinear form is coervice only, then the Fredholm alternative is needed to prove the existence of the solution.

3.1. Theorem (*Lax-Milgram*). Let $B(u,v)$ be a bilinear form on a Hilbert space V with inner product $(\cdot,\cdot)_V$ and norm $\|\cdot\|_V$, and assume that B is *bounded* on V, i.e. there is a constant $k \geq 0$ such that for all $u,v \in V$

$$|B(u,v)| \leq k \; \|u\|_V \; \|v\|_V,$$

and that B is *strongly coercive* over V. Then every bounded linear functional $F(v)$ on V can be represented in the form

$$F(v) = B(u,v) \tag{3.1}$$

for some $u \in V$, which is uniquely determined by F.

Proof: See Friedman [1969], p. 41. □

This theorem is a generalization of the *Riesz representation theorem* and is in agreement with the remark made before, that a strongly coercive bilinear form resembles an inner product.

3.2. Corollary. Applying the Riesz representation theorem to (3.1) we can define a mapping $A:V \to V$ by

$$B(u,v) = (Au,v)_V.$$

It can be shown that this mapping A is linear and bounded, that A is surjective, so that A has a bounded inverse A^{-1}. See e.g. Friedman [1969], p. 41 ff. Consequently, there exists for each $g \in V$ a uniquely determined $u \in V$ solving the equation

$$(Au,v)_V = (g,v)_V \quad \text{for all } v \in V,$$

which is equivalent to

$$Au = g$$

considered as an abstract operator equation in the Hilbert space V.

The following existence theorem is a direct consequence of the Lax-Milgram theorem.

3.3. Theorem. Assume that the uniformly strongly elliptic operator L has an associated bilinear form that is strongly coercive over $H_0^m(\Omega)$. Then there exists a unique solution of the generalized Dirichlet problem with homogeneous boundary data

(3.2) $$Lu = f \quad \text{in } \Omega,\ f \in L^2(\Omega),$$

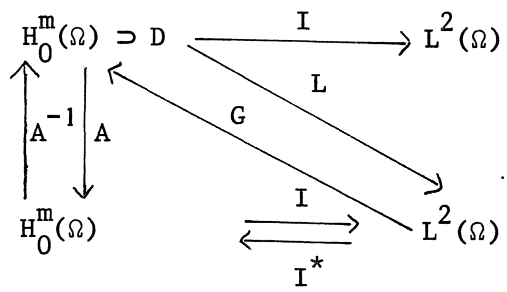
(3.3) $$u = 0 \quad \text{on } \partial\Omega.$$

Proof: See also Friedman [1969], p. 43. $H_0^m(\Omega)$ plays the role of V mentioned in 3.2, and $(f,v)_{0,\Omega}$ the role of F(v) in theorem 3.1. □

Possibly the following diagram can be of help in understanding the situation. The differential operator L is an unbounded operator from the subset $D = \{v \in H_0^m(\Omega) \mid Lv \in L^2(\Omega)\}$ onto $L^2(\Omega)$, whereas A is a bounded mapping from $H_0^m(\Omega)$ onto itself; I is the bounded imbedding discussed at the end of section 2. The bounded mapping I^* is defined by the equality (again the Riesz representation theorem is used)

$$(z,Iv)_{0,\Omega} = (I^*z,v)_{m,\Omega},$$

where $z \in L^2(\Omega)$, $v \in H_0^m(\Omega)$. Given a $f \in L^2(\Omega)$, the solution of the generalized Dirichlet problem with homogeneous boundary data (3.2), (3.3) can then be expressed as $u = Gf \overset{\text{def}}{=} A^{-1}I^*f$; from the definition it follows that G is bounded.



3.4. Corollary. Assume that the operator L satisfies (L1), (L2) and (L3) of theorem 2.15, and let λ_0 be the constant appearing in (2.28). Then the generalized Dirichlet problem with homogeneous boundary data for the equation $(L+\lambda)u = f$ has a unique solution for any $\lambda \geq \lambda_0$.

Proof: The bilinear form associated with $L + \lambda$, i.e.

$$B_\lambda(u,v) = B(u,v) + \lambda(u,v)_{0,\Omega},$$

is strongly coercive over $H_0^m(\Omega)$. □

What remains is the investigation of the generalized Dirichlet problem for $L + \lambda$, with $\lambda < \lambda_0$, the case in which $B_\lambda(u,v)$ is not *strongly* coercive over $H_0^m(\Omega)$. Without loss of generality we assume $\lambda = 0$. First the Dirichlet problem for $L + \lambda_0$ is considered in the same fashion as in theorem 3.3 and 3.4. The mapping A mentioned in corollary 3.2 is now defined by

$$B_{\lambda_0}(u,v) = (Au,v)_{m,\Omega}.$$

The maps A^{-1} and G are introduced consistently with this definition. From theorem 1.11 we know that the identity mapping

$$I: H_0^m(\Omega) \to L^2(\Omega)$$

is compact. Consequently, the composition

$$IG: L^2(\Omega) \to L^2(\Omega)$$

is also compact, for G is a bounded mapping. Having a compact mapping of $L^2(\Omega)$ into itself, we may apply the Riesz-Schander theory. The following theorem is based on this theory. For details see Agmon [1965], p. 102 ff., and Friedman [1969], p. 44 ff.

3.5. Theorem. Let L satisfy the conditions (L1), (L2) and (L3) (see theorem 2.15). Then the Fredholm alternative holds for the generalized Dirichlet problem with homogeneous boundary data. More precisely, if B is the bilinear form associated with L, then for any $f \in L^2(\Omega)$ either there exists a unique solution $u \in H_0^m(\Omega)$ of

$$(3.4) \qquad B(u,v) = (f,v)_{0,\Omega} \qquad \text{for all } v \in H_0^m(\Omega),$$

or the homogeneous equation

$$(3.5) \qquad B(v,u) = 0 \qquad \text{for all } v \in H_0^m(\Omega)$$

has a finite number of linearly independent solutions u_j, $j=1,\ldots,k$, belonging to $H_0^m(\Omega)$, in which case there exists a solution of (3.4) if and only if

$$(f,u_j)_{0,\Omega} = 0, \qquad j=1,\ldots,k;$$

this solution is not unique.

Proof: Apply the Fredholm alternative to the equation

$$u - \lambda_0 IGu = IGf,$$

which is equivalent to (3.4), G being the operator defined in 3.4. □

3.6. Remark. It is easily seen that the complex conjugate of B(v,u) is the bilinear form associated with the adjoint L^* of L.

4. REGULARITY OF SOLUTIONS

Consider a solution $u \in H^m(\Omega)$ of the generalized Dirichlet problem. In regularity theory the question is dealt with, which conditions should be imposed on the bilinear form, such that u belongs to a higher order Sobolev space $H^{m+k}(\Omega)$. This question is important for the following reason. Sobolev's theorem shows that, given a non-negative integer k, for sufficiently large j the Sobolev space of order j is imbedded in the space of k times continuously differentiable functions $C^k(\Omega)$. If a generalized solution is to be classical, it therefore has to belong to such a Sobolev space.

Throughout this section L will be the differential operator of order 2m given in definition 2.10, and B the bilinear form associated with L.

4.1. Global outline. We shall try to give the reader a global idea along which lines the argument of the regularity theory runs. For details one is referred to e.g. Friedman [1969], p. 47 ff.

The regularity theory is divided into two parts, the first dealing with regularity of the solution in the interior of the domain Ω, the second handling the regularity up to the boundary $\partial\Omega$.

The steps to prove the regularity in the interior are the following. First we observe that the difference quotient with respect to x_i, $i=1,\ldots,n$,

$$(4.1) \qquad u^k(x) = h^{-1}(u(x+he^i) - u(x)),$$

where $e^i = (\delta_{i1},\ldots,\delta_{in})$, δ_{ik} being the Kronecker delta, is well defined in each subdomain Ω' with $\bar{\Omega}' \subset \Omega$, if h is smaller than the distance of Ω' to $\partial\Omega$. If $u \in H^k(\Omega)$, then of course $u^h \in H^k(\Omega')$ for any Ω' with $\bar{\Omega}' \subset \Omega$. Furthermore, it can be shown that if $u \in H^k(\Omega)$, then $\lim\limits_{h\to 0} \|u^h - D_i u\|_{k-1,\Omega'}$, and also that if $u \in H^k(\Omega)$ and if for any subdomain Ω' with $\bar{\Omega}' \subset \Omega$, $\|u^h\|_{k,\Omega'} \leq$ $\leq c$ for all sufficiently small h, where c is a constant independent of Ω' and h, then $D_i u \in H^k(\Omega)$ and $\|D_i u\|_{k,\Omega} \leq c$.

The next step to establish interior regularity of the solution is to prove that under certain assumptions for the bilinear form B associated with L, which will be specified below, the difference quotient u^h of a function $u \in H^m(\Omega)$ satisfying

$$B(u,v) = (f,v)_{0,\Omega} \qquad \text{for all } v \in H_0^m(\Omega)$$

is uniformly bounded for sufficiently small h. By passing to the limit $h \to 0$ then by means of the result stated above it is proved that $D_i u \in H^m(\Omega)$, $i=1,\dots,n$, from which it follows that $u \in H^{m+1}(\Omega)$. This line of reasoning can be repeated a number of times, depending on the properties of B.

We now simply state the final results for interior regularity.

4.2. Definition. Let j be a non-negative integer. The bilinear form

$$B(u,v) = \sum_{|p|,|q|\le m} (a_{pq}(x)D^q u, D^p u)_{0,\Omega}$$

is said to be *j-smooth* if the coefficients $a_{pq}(x)$, for $|p|+j-m \ge 0$, $|q| \le m$, belong to $C^{|p|+j-m}(\bar{\Omega})$.

If this assumption holds, then we denote by K a bound on the first $|p|+j-m$ derivatives of all coefficients $a_{pq}(x)$ in Ω where $|p|+j-m \ge 0$, $|q| \le m$.

4.3. Theorem. Let $1 \le j \le m$. Assume that

(L1) L is uniformly strongly elliptic in Ω with ellipticity module E;

(L2) the coefficients $a_{pq}(x)$ are uniformly bounded by a constant M;

(S_j) the bilinear form B associated with L is j-smooth.

Let further $f \in L^2(\Omega)$, and let $u \in H^m(\Omega)$ satisfy the equation

$$B(u,v) = (f,v)_{0,\Omega} \qquad \text{for all } v \in H_0^m(\Omega). \tag{4.1}$$

Then for any subdomain Ω' of Ω with $\bar{\Omega}' \subset \Omega$, u belongs to $H^{m+j}(\Omega')$, and the inequality

$$\|u\|_{m+j,\Omega'} \le c\,(\|f\|_{0,\Omega} + \|u\|_{m,\Omega}) \tag{4.2}$$

holds; here c is a constant depending only on E, M, K, Ω and Ω'.

Proof: See Friedman [1969], p. 46 ff., or Agmon [1965], p. 51 ff. □

4.4. Theorem (*regularity in the interior*). Let the assumptions (L1), (L2) and (S_{m+k}) (i.e., B is (m+k)-smooth) hold for some $k \geq 0$, and let $f \in H^k(\Omega)$. If $u \in H^m(\Omega)$ satisfies equation (5.1), then $u \in H^{2m+k}(\Omega')$ for any subdomain Ω' with $\bar{\Omega}' \subset \Omega$ and

$$\|u\|_{2m+k,\Omega'} \leq c\ (\|f\|_{k,\Omega} + \|u\|_{m,\Omega}), \tag{4.3}$$

c depending only on E, M, K, Ω and Ω'.

Proof: By induction on k. □

Turning our attention to regularity up to the boundary, we first state the following lemma.

4.5. Lemma. Let $B^n_+(R)$ denote the half-ball $\{x \in \mathbb{R}^n \mid |x| < R,\ x_n > 0\}$. Then theorems 4.3 and 4.4 remain true for $\Omega = B^n_+(R)$ and $\Omega' = B^n_+(r)$, where $r < R$.

Proof: See Friedman [1969], p. 61 ff., or Agmon [1965], p. 103 ff. □

This lemma gives us regularity up to the "flat" part of the boundary of the half-ball. The idea of the proof is that the partial derivatives with respect to x_n, $D^j_n u$, can be expressed in terms of the other partial derivatives by making use of the fact that u satisfies equation (4.1) in $B^n_+(R)$. Consequently, derivatives along the normal at the "flat" part of the boundary can be replaced by derivatives along the boundary; hence regularity up to the part of the boundary that coincides with $\mathbb{R}^{n-1}$ is established.

Global regularity for an arbitrary bounded region with sufficiently smooth boundary is proved in the following manner. Cover the boundary $\partial\Omega$ by a finite number of open bounded regions Ω_i in such a way that $\Omega_i \cap \Omega$ can be mapped one-to-one onto a half-ball $B^n_+(Ri)$. According to definition 1.6 these maps and their inverses are of the same differentiability as the boundary $\partial\Omega$. By taking a suitable open set Ω_0 with $\bar{\Omega}_0 \subset \Omega$, $\bar{\Omega}$ is completely covered. Application of theorem 4.4 in Ω_0 and of lemma 4.5 then entails the following theorem on global regularity. For details, again, see Friedman [1969]. In the proof the partition of unity is needed.

4.6. Theorem. Let Ω be a bounded region with $\partial\Omega$ of class C^{2m}, and let assumptions (L1), (L2) and (S_j) hold for some j, $1 \le j \le m$. Let further $f \in L^2(\Omega)$, and let $u \in H_0^m(\Omega)$ solve (4.1). Then u belongs to $H^{m+j}(\Omega)$, and inequality (4.2) holds with Ω' replaced by Ω.

If, for some $k \ge 0$, $\partial\Omega$ is of class C^{2m+k}, if $f \in H^k(\Omega)$, and if (S_{m+k}) holds, i.e. $a_{pq}(x) \in C^{|p|+k}(\bar{\Omega})$ for all $|p|,|q| \le m$, then $u \in H^{2m+k}(\Omega)$, and inequality (4.3) is valid with Ω' replaced by Ω.

By combining theorem 4.4 and Sobolev's theorem, we have:

4.7. <u>Theorem</u>. Let Ω be a bounded region in $\mathbb{R}^n$, and let L be a uniformly strongly elliptic operator in Ω satisfying (L1) and (L2). Suppose further that for some $k \ge 0$, but satisfying $2m+k > [n/2]$, the bilinear form B associated with L is (m+k)-smooth and the right hand side of the equation $f \in H^k(\Omega)$. Then the following assertion is true:

If u is a generalized solution of the differential equation $Lu = f$ in Ω, i.e. if u solves (4.1), then u belongs to $C^\ell(\Omega)$, where $\ell = 2m+k-[n/2]-1$. More precisely, u can then be redefined on a set of measure zero such that $u \in C^\ell(\Omega)$.

This theorem tells us in which case a generalized solution of the differential equation is also a classical solution. What remains is the question when a solution u satisfying the boundary conditions in the generalized sense (see definition 1.10) also satisfies these conditions in the classical sense, i.e. $u \in C^{m-1}(\bar{\Omega})$ for the Dirichlet problem. The answer is given by the following theorem, which is a consequence of theorem 4.6 and Sobolev's theorem.

4.8. <u>Theorem</u>. Let Ω, L, B, f and k be as in the preceding theorem. Suppose additionally, that Ω has a boundary of class C^{2m+k}. If u is a generalized solution of the Dirichlet problem, then it can be redefined on a subset of Ω of measure zero such that $u \in C^\ell(\bar{\Omega})$, where $\ell = 2m+k-[n/2]-1$.

4.9. <u>Remark</u>. If, in particular, the coefficients of L belong to $C^\infty(\bar{\Omega})$, if $f \in C^\infty(\bar{\Omega})$ and if the boundary of Ω is of class C^∞, then the generalized solution of the Dirichlet problem is infinitely differentiable in $\bar{\Omega}$.

5. OTHER BOUNDARY CONDITIONS

So far only Dirichlet boundary conditions have been considered. In this section it is shown how some other boundary conditions are dealt with. As opposed to the Dirichlet problem, that allows a uniform treatment for equations of any order 2m, it is far more difficult to handle other boundary conditions for equations of order 2m, $m \geq 2$, than for order 2. The reason for this difference is to be found in the fact that in the case $m = 1$ the bilinear form associated with the elliptic differential operator, as given in section 2, is (strongly) coercive over $H^1(\Omega)$, where in the case $m \geq 2$ this bilinear form is only coercive over the subspace $H_0^m(\Omega)$ of $H^m(\Omega)$. Here we shall restrict ourselves to second order equations.

Let L be the uniformly strongly elliptic operator of order 2 defined in (2.1) and B the bilinear form associated with L, given in (2.7). In this section we shall not bother about regularity, and assume that $u \in H^1(\Omega)$ and $Lu \in L^2(\Omega)$. The regularity theory of the preceding section is easily applied to the examples that follow. The following version of Green's identity is useful.

5.1. Green's identity. For sufficiently smooth $\partial\Omega$, a_{ij} $(i,j=1,\dots,n)$, a_i $(i=0,\dots,n)$, u and v the following equality holds:

$$(5.1) \qquad B(u,v) = (Lu,v)_{0,\Omega} + \int_{\partial\Omega} \sum_{i,j=1}^{n} \bar{v}\, a_{ij} \frac{\partial x_i}{\partial \nu} D_j u \, d\sigma,$$

where $d\sigma$ is a surface (line) element of $\partial\Omega$.

5.2. Remark. If V is a closed subspace of $H^1(\Omega)$, such that $H_0^1(\Omega) \subset V$, and if B is a (strongly) coercive bilinear form over $H^1(\Omega)$, it is also (strongly) coercive over any such V. In the following examples we make different choices for V. An interpretation for these choices of V will be given for the problem of finding a $u \in V$ such that

$$B(u,v) = (f,v)_{0,\Omega} \qquad \text{for all } v \in V$$

holds. The unique existence of such a function u follows from the considerations made in section 3.

5.3. Dirichlet problem. If we take in (5.1) $u,v \in V = H_0^1(\Omega)$, then we have

$$B(u,v) = (Lu,v)_{0,\Omega} \qquad \text{for all } v \in H_0^1(\Omega).$$

Further, in accordance with section 3, there exists one $u \in H_0^1(\Omega)$ satisfying

$$B(u,v) = (f,v)_{0,\Omega} \qquad \text{for all } v \in H_0^1(\Omega).$$

Consequently, for every $f \in L^2(\Omega)$ there exists a uniquely determined generalized solution of

(5.2) $\qquad Lu = f$ in Ω, $\qquad u = 0$ on $\partial\Omega$.

5.4. Neumann problem. Let $V = H^1(\Omega)$, then from Green's identity it follows that

$$\int_{\partial\Omega} \bar{v} \sum_{i,j=1}^{n} a_{ij} \frac{\partial x_i}{\partial\Omega} D_j u \, d\sigma = 0 \qquad \text{for all } v \in H^1(\Omega).$$

This identity can only hold if the expression

$$\frac{\partial u}{\partial \nu_L} \overset{\text{def}}{=} \sum_{i,j=1}^{n} a_{ij} \frac{\partial x_i}{\partial \nu} D_j u = 0.$$

So for each $f \in L^2(\Omega)$ there is a unique generalized solution of the problem

(5.3) $\qquad Lu = f$ in Ω, $\dfrac{\partial u}{\partial \nu_L} = 0$ on $\partial\Omega$,

which is known as the Neumann problem.

Notice that in the case that L is the Laplace operator $-\Delta$, $\partial u/\partial\nu_L = \partial u/\partial\nu$, the derivative taken along the normal to the boundary.

5.5. <u>Mixed problem</u>. Let $\partial_1\Omega$ be an open subset of $\partial\Omega$, and let $\partial_2\Omega = \partial\Omega\backslash\partial_1\Omega$. Choose V to be the completion in $H^1(\Omega)$ of the set of functions $\phi \in C^\infty(\bar{\Omega})$ that are zero in a certain neighbourhood of $\partial_1\Omega$. Then V can be interpreted as the closed linear subspace of all functions $H^1(\Omega)$ that are zero on $\partial_1\Omega$, for which only a slight modification of lemma 1.9 is needed. Application of the existence theory to B considered as a bilinear form on V yields the unique existence of a generalized solution of the mixed boundary value problem

(5.4) $$Lu = f \text{ in } \Omega,\ u = 0 \text{ on } \partial_1\Omega,\ \partial u/\partial\nu_L = 0 \text{ on } \partial_2\Omega.$$

6. REGULARITY IN DOMAINS WITH NON-SMOOTH BOUNDARIES

In section 4 global regularity of a solution has only been considered in the case that the domain Ω, in which regularity has to be demonstrated, fulfills rather strong smoothness requirements. The question of regularity of the solution of the Dirichlet problem for the equation $Lu = f$ in a domain with "corners", of which the rectangle is an example, cannot be answered in a general manner. Different situations have to be studied in a separate way. The book of Nečas [1967] is the only one among the books mentioned in the bibliography at the end of this report, which deals with this subject at some length. More or less as an illustration, two results for 2- and 3-dimensional domains respectively, will be reproduced here, the formulation of which is relatively simple, though the proofs are not.

6.1. <u>Definition</u>. A domain $\Omega \subset \mathbb{R}^n$ is said to satisfy the *exterior cone condition*, if there exists a fixed cone C with a certain given opening, such that for each point $x_0 \in \partial\Omega$ one can place the cone with its top in x_0 and further completely outside Ω; see figure 6.1.

A domain has the *exterior ball property*, if there exists a fixed ball B, such that for each $x_0 \in \partial\Omega$ the ball can be placed completely outside Ω with $x_0 \in \partial B$, see figure 6.2.

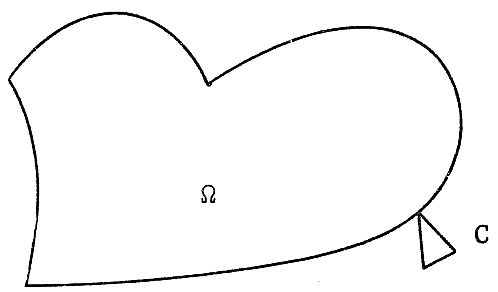


Figure 6.1

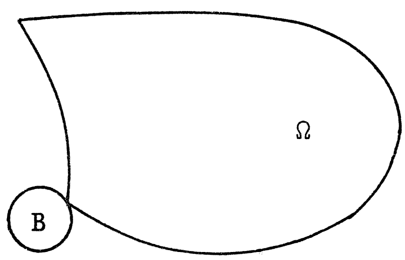


Figure 6.2

Note that a domain in the plane having a Lipschitz continuous boundary satisfies the exterior cone condition.

6.2. <u>Theorem</u>. Let Ω be a bounded domain in the plane $\mathbb{R}^2$ having a Lipschitz continuous boundary. Let L be an elliptic operator of order 2m, satisfying the conditions (L1), (L2) and (L3). Let also the coefficients of L satisfy the smoothness requirement

$$a_{pq} \in C^{|p|-m+1}(\bar{\Omega})$$

for $|p|-m+1 \geq 0$, $|q| \leq m$.

If $u \in H_0^m(\Omega)$ is the solution of the generalized Dirichlet problem with homogeneous boundary data for $Lu = f$, $f \in L^2(\Omega)$, then $u \in H^{m+1}(\Omega)$ and

(6.2) $$||u||_{m+1,\Omega} \leq \text{const.} \, (||f||_{0,\Omega} + ||u||_{m,\Omega})$$

where the constant is some positive number.

Proof: See Nečas [1967], p. 326 f. □

Nečas does not state the result in the above form. We have chosen this formulation to make it consistent with theorem 4.6 and inequality (4.2).

6.3. <u>Corollary</u>. Let the same Dirichlet problem be given as in theorem 6.2, but with (6.1) replaced by

$$a_{pq} \in C^{|p|+2}(\bar{\Omega}), \qquad |p|,|q| \leq m,$$

and with $f \in H^2(\Omega)$ instead of $L^2(\Omega)$. Then the generalized solution of this Dirichlet problem can be redefined on a set of measure zero, such that u is a classical solution, i.e. $u \in C^{m-1}(\bar{\Omega}) \cap C^{2m}(\Omega)$.

Proof: Note that if $f \in H^2(\Omega)$, we may consider f to belong to $C^0(\bar{\Omega})$, according to Sobolev's theorem (theorem 1.13). From theorem 4.4 it follows that $u \in H^{2m+2}(\Omega')$ for any subdomain Ω' of Ω with $\bar{\Omega}' \subset \Omega$. Again as a consequence of Sobolev's theorem we find for domains in $\mathbb{R}^2$ $H^{2m+2}(\Omega') \subset C^{2m}(\Omega')$ for all Ω' with $\bar{\Omega}' \subset \Omega$, thus $u \in C^{2m}(\Omega)$. From theorem 6.2 we infer that $u \in C^{m-1}(\bar{\Omega})$, since $H^{m+1}(\Omega) \subset C^{m+1}(\bar{\Omega})$. □

6.4. <u>Remark</u>. If the bilinear form B associated with the differential operator L is strongly coercive over $H_0^m(\Omega)$, and if u is the solution of the generalized Dirichlet problem with homogeneous boundary data, then u satisfies the inequalities

$$cE\|u\|_m^2 \le \text{Re } B(u,u) = \text{Re}(f,u)_0 \le \|f\|_0 \ \|u\|_0,$$

where c is some positive constant. Consequently, $\|u\|_m$ is bounded from above by $c_1\|f\|_0$, where c_1 is some positive constant. Substitution of this upperbound into (6.2) leads to a reformulation of the assertion of theorem 6.2, namely

$$(6.3) \qquad \|u\|_{m+1,\Omega} \le \text{const. } \|f\|_{0,\Omega}.$$

where the constant is a positive number.

We now consider the case of a 3-dimensional domain.

6.4. <u>Theorem</u>. Let Ω be a bounded domain in $\mathbb{R}^3$ with a Lipschitz continuous boundary, and satisfying the exterior ball condition. Let L be an elliptic operator in Ω, satisfying the same conditions as the operator L in theorem 6.2. Suppose, additionally, that L has real coefficients and that the bilinear form associated with L is strongly coercive. If $u \in H_0^m(\Omega)$ solves the equation $Lu = f$, $f \in L^2(\Omega)$, in the generalized sense, then $u \in C^{k-1}(\bar{\Omega})$, and

$$\sum_{|p|\le k-1} \sup_{\Omega} |u(x)| \le \text{const. } \|f\|_{0,\Omega} \ .$$

Proof: The proof is quite complicated, and involves concepts that have not been mentioned in this report. See Nečas [1967], Ch. 6 and 7. □

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